Analytical Solutions for the Developing Jet From a Fully-Developed Laminar Tube Flow

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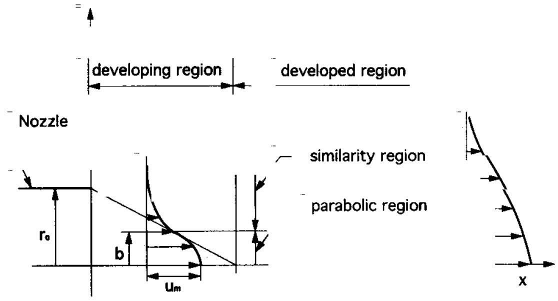

Fig. 1 An axisymmetric laminar jet developing from the parabolic exit velocity profile

Introduction

The very classical problem of the developing jet exiting from a tube flow (Fig. 1) is of particular interest to those involved with gaseous or particulate fuel injection as an initial condition for the successive combustion analysis. The fully developed flow field farther away from the tube exit is generally well described by the Schlichting's similarity solution (Schlichting, 1968) . However, the similarity assumption fails to hold in the developing region of the jet close to the exit. Conventional analysis of the developing region numerically solves the Prandtl's nonlinear boundary layer equations for detailed velocity profiles (Pai and Hsieh, 1972; du Plessis et al., 1973; Dmitriev and Kulesova, 1974; Arulraja, 1982). Though most of the numerical solutions agree reasonably well with the existing experimental data (Okabe, 1948; Abramovich and Solan, 1973; Rankin et al., 1983) achieving an analytical solution will be of an important value based on its closed-form comprehensiveness and physically intuitive nature.

Arulraja et al. (1983) derived an analytical solution only for the maximum center velocity decay along the jet axis. The results show that the maximum velocity decays linearly in the developing region. Matching of the maximum velocity at the boundary between the developing and fully developed similarity regions determined the length of the developing region.

For the simpler case of uniform nozzle exit velocity profile, Rankin and Sridhar (1978) developed an analytical solution for the velocity profiles in the developing region assuming a potential core of uniform velocity. In their analysis, the surrounding region was assumed as annular free shear flow holding the Schlichting's similarity velocity profile.

The current work proposes two approximate methods to analytically calculate the developing jet velocity field from a fully developed laminar (parabolic) axisymmetric tube flow.

Method I: Linearization of the Boundary Layer Equations

The Prandtl's boundary layer equation, a nonlinear secondorder parabolic partial differential type (Schlichting, 1968) describes the axisymmetric laminar jet flow lield (Fig. 1).

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u}{\partial r}\right] \quad (1)$$

in which the total momentum remains unchanged along the jet, the total mass flow rate gradually increases because of the entrainment from the ambiance.

Three assumptions are made particularly for the jet developing region near the tube exit: (1) the relatively short jet evolving distance negates the entrainment effect, (2) the linear radial diffusion dominates over the nonlinear axial convection, and (3) the Lagrangian particle motion along the centerline describes the axial convection. With these assumptions the nonlinear boundary layer equation, Eq. (1), can be decoupled into two linear differential equations, i.e., one is a diffusion type equation that describes the radial diffusion in the absence of entrainment,

$$\frac{\partial u(t,r)}{\partial t} = \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u(t,r)}{\partial r}\right] \quad (2)$$

and the other is a Lagrangian description of fluid particle motion along the centerline, which substitutes the nonlinear convection,

$$\frac{dx}{dt} = u_m(x) \quad (3)$$

where the subscript m denotes the maximum or centerline velocity.

The boundary and initial conditions are

$$\frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0 \quad \text{and} \quad u \to 0 \quad \text{as} \quad r \to \infty \quad (4a, b)$$

$$u(r, t = 0) = u_{m,o}[1 - (r/r_o)^2] \quad \text{for} \quad 0 \le r \le r_o \quad (4c)$$

where $u_{m,o} = u_m\ (x = 0)$, the maximum centerline velocity at the tube exit, and ro is the nozzle radius. Solving Eq. (2) together with Eqs. (4*a, b, c*) gives

$$\frac{u(t^*, R)}{u_{m,o}} = 2t^*e^{-R^2t^*}\int_0^1 \eta(1-\eta^2)e^{-\eta^2t^*}I_0(2\eta Rt^*)d\eta \quad (5)$$

where t^* is a dimensionless Lagrangian time defined as $r^o/4\nu t$, $R \equiv r/r_o$, and to is the zeroth-order modified Bessel function of the first kind.

Integration of Eq. (3) using the initial condition of Eq. (4c) provides a relationship between the Eulerian axial coordinate X and the Lagrangian time coordinate t^* as

$$X \equiv \frac{x}{d_o}\cdot\frac{\nu}{u_{m,o}d_o} = \frac{1}{16}\int_0^{1/t^*}[1 - \eta(1 - e^{-1/\eta})]d\eta \quad (6)$$

where d_o denotes the nozzle diameter.

Substituting Eq. (6) into Eq. (5) determines the jet velocity profiles *u(X, R)*, or *u(x, r)*, for the developing region-this solution is called near-field solution where the flow similarity is not established. The well-known Schlichting's similarity solution

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(1968) constitutes far-field solution for the developed region, where the flow similarity prevails.

Method II: Velocity Profile Matching

The experimental data of Rankin et al. (1983) indicate that the jet core of a certain radial distance in the developing region maintains approximately the initial parabolic profile, and this "parabolic core" region diminishes as the similarity velocity profile prevails in the annular outer region (Fig. 1). The radius of the parabolic core decreases to zero as the developing region ends at X = Xl where X, denotes the developing regional length.

$$U_{\text{inner}}(X, R) = \left(\frac{u(X, R)}{u_{m,o}}\right)_{\text{inner}} = \frac{u_m(X)}{u_{m,o}}[1 - R^2] \quad \text{for} \quad 0 \le R \le B \qquad (7a)$$

and for the outer region (Schlichting, 1968)

$$U_{\text{outer}}(X, R) = \left(\frac{u(X, R)}{u_{m,o}}\right)_{\text{outer}} = \frac{2(\gamma/\text{Re}_{d_o})^2}{(X + X_v)} \times \left[1 + \left(\frac{\gamma}{4\,\text{Re}_{d_o}}\frac{R}{(X + X_v)}\right)^2\right]^{-2} \quad \text{for } B \le R \le \infty \qquad (7b)$$

where $B(X)$ denotes the core radius normalized by the nozzle radius, i.e., $b(x)/r_o$, X_v is the location of virtual origin measured from the nozzle exit, and the dimensionless radius $R \equiv r/r_o$.

An interpolation of the maximum center velocity calculated from Method I gives

$$\frac{u_m(X)}{u_{m,o}} = 1 - 17.72X \quad \text{for} \quad 0 \le X \le X_l \qquad (8)$$

with a confidence of multiple determination $R^2 = 0.99$.

The jet velocity profiles, Eqs. (7a) and (7b), must be continuous and smooth at the regional boundary $R = B$;

$$U_{\text{inner}} = U_{\text{outer}} \quad \text{and} \quad \left[\frac{\partial U}{\partial R}\right]_{\text{inner}} = \left[\frac{\partial U}{\partial R}\right]_{\text{outer}} \quad \text{at} \quad R = B \qquad (9a, b)$$

In addition, the axial momentum of the free shear axisymmetric incompressible jet must be conserved:

$$\frac{d}{dx}\int_0^\infty RU^2 dR = 0 \qquad (10)$$

where U = U_{inner} (Eq. 7a) for $0 \le R \le B$, $U = U_{outer}$, (Eq. 7b) for $B \le R \le \infty$, and $U = 1 - R^2$ at $X = 0$.

Solving Eqs. (9a, b) and (10) determines the inner core radius B.

$$B = \sqrt{1 - \left\{\frac{1}{(1 - 17.72X)^2} - 1\right\}^{1/3}} \qquad (11)$$

the location of the virtual origin,

$$X_v = -X + \frac{1 - 17.72X}{16} \cdot \frac{(1 - B^2)}{1 - 1.5B^2} \qquad (12)$$

and the jet spread parameter

$$\frac{\gamma}{\text{Re}_{d_o}} = \frac{1 - 17.72X}{2^{2.5}}\left(\frac{1 - B^4}{\sqrt{1 - 1.5B^2}}\right) \qquad (13)$$

Substitution of Eqs. (1 I) through (13) back into Eqs. (7a) and (7b)determjnes the jet velocity distribution $U(X, R)$ for the whole developing region. The length of the inner core, X, = 0.0165, is readily calculated by setting the core radius B equal to zero in Eq. (11) .

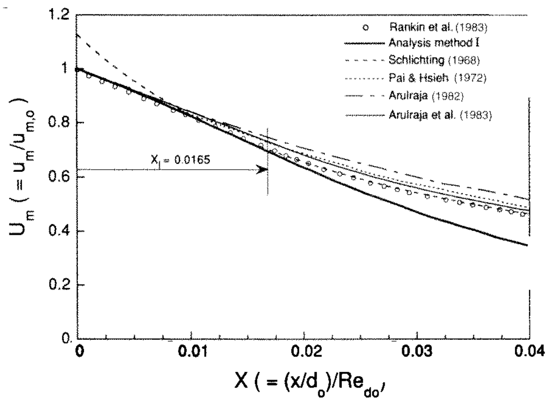

Fig. 2 Comparison of maximum velocity variations along the jet axial distance

Comparison With Experimental Results

Figure 2 presents the axial decay of the centerline velocity normalized by $u_{m,0}$ at the tube exit. For the developing region, $X < X_l$, the linearized analysis of Method I shows better agreement with the experimental data (Rankin et al., 1983) than the other numerical (Pai and Hsieh, 1972) and analytical results (Arulraja, 1982, Arulraja et al., 1983). The Schlichting's similarity solution deviates from the experiment in the developing region which lacks the flow similarity.

In the developed region, $X > X_l$, the center line velocity calculations from Method I underpredict compared with the experimental data. This deficiency comes primarily from the fact that Method I does not account for the flow entrainment. The effect of entrainment remains small in the developing region due to the shorter jet travel distance. When X is large, however, the gradual addition of the entrained flow enhances the total flow rate, and in turn, the centerline velocity is greater than the Method I prediction given under no entrainment. The similarity solution, which accounts for the entrainment, well predicts the centerline velocity variation when $X > X_l$.

Figure 3 presents the jet velocity profiles calculated from both Method I and Method II along X inside the developing region, compared with experiment (Rankin et al., 1983) and the similarity solutions (Schlichting, 1968) . Near the nozzle exit ($X = 0.001$), both Method I and Method II predict fairly well while the Schlichting's similarity solution gives large discrepancies from the experimental data particularly at the center and near the jet boundary. The linearized approximation of Method I is fairly well justified in this region close to the nozzle exit where the jet develops with negligibly small entrainment.

As X increases ($X = 0.005$ and 0.01), the entrainment effect is no longer negligible and Method I underpredicts since its result does not account for the entrainment. In practice, the entrainment effect will gradually increase the jet flow rate and expand the jet radius as the jet develops. This can explain the underprediction of Method I and the deficiency increases with increasing axial distance.

Method II analysis is exclusively based on the velocity matching between the near (parabolic) and far (similar) fields. Therefore, Method II retains the entrainment effect by its nature and the calculated velocity profiles show excellent agreement with the experimental data for all the tested X locations inside the developing region. Although not shown, the maximum centerline velocities calculated from Method II also show excellent agreement with experiment.

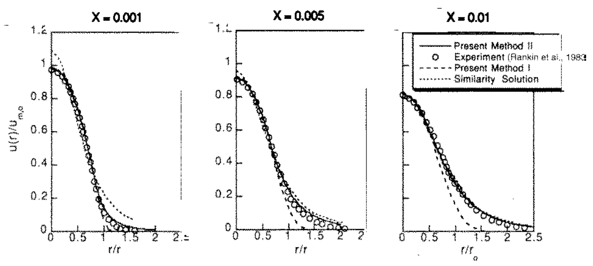

Fig. 3 Comparison of the axial jet velocity profiles calculated from Method I and Method II with experimental data [8] and the Schlichting's similarity solution [1]

In summary, both Methods I and II accurately calculate the maximum centerline velocity decay for the jet developing region, but Method II more accurately predicts the jet velocity fields in detail. Outside the developing region, the Schlichting's similarity solution is well accepted.

Acknowledgment

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